Recall:

$$\mathrm{Ind}(D_t) = \int_X \mathrm{Tr}_s^E(P_t(x,x))\,dv(x)$$
$$= t^{-m/2}\sum_{j=0}^{N} t^j \int_X \mathrm{Tr}_s^E[a_j(x)]\,dv(x) + O(t^{-\frac{m}{2}+N+1})$$

Thm:

$$\mathrm{Tr}_s^E[a_j(x)]\,dv(x) = \begin{cases} 0 & j < \frac{m}{2} \\ \left[\hat{A}(TX,\nabla^{TX})\,\mathrm{ch}^{E/S}(E,\nabla^E)\right]^{\max} & j = \frac{m}{2} \end{cases}$$

Step 1: $u > 0$

$$e^{-\frac{uD^2}{2}}(x,x) = F_u(\sqrt{u}D)(x,x) + O(e^{-c/u})$$

where
$$F_u(\lambda) = \int_{-\infty}^{+\infty} \cos(s\lambda)\, e^{-s^2/2} f(\sqrt{u}s)\frac{ds}{\sqrt{2\pi}}$$

$f \in C^\infty(\mathbb{R},[0,1])$ even fct

$$f(s) = \begin{cases} 1 & |s| \leq \varepsilon/2 \\ 0 & |s| \geq \varepsilon \end{cases}$$

$F_u(\sqrt{u}D)(x,x)$ depends only on $D^2|_{B^X(x_0,\varepsilon)}$

Step 2: $X_0 = T_{x_0}X \quad g^{TX_0}$

$X \supset B^X(x_0,4\varepsilon) \qquad B^{X_0}(0,4\varepsilon)$

$D^2,\ g^{TX},\ E,\ \nabla^E,\ h^E \qquad L,\ g^{TX_0},\ E_0,\ \nabla^{E_0},\ h^{E_0}$

on $B^X(x_0, 2\varepsilon) \simeq B^{X_0}(0, 2\varepsilon)$

we identify both sides

For the point $Z \in B^{X_0}(0, 4\varepsilon)^c$

$$g^{TX_0} = g^{TX}_{x_0}$$

$$E_0 = E_{x_0}$$

$$\nabla^{E_0} = d$$

$$h^{E_0} = h^E_{x_0}$$

Cut-off function $\rho(Z) = \begin{cases} 1 & |Z| \leq 2\varepsilon \\ 0 & |Z| \geq 4\varepsilon \end{cases}$

$$L = \Delta^{E_0} + \rho(Z)\left(\frac{r^X}{4} + \frac{1}{2}\sum_{j,k} R^W(\tilde{e}_j, \tilde{e}_k)c(\tilde{e}_j)c(\tilde{e}_k)\right)$$

$$L|_{B^{X_0}(0,2\varepsilon)} = D^2|_{B^X(x_0, 2\varepsilon)}$$

$$L|_{B^{X_0}(0,4\varepsilon)^c} = -\sum_j \left(\frac{\partial}{\partial Z_j}\right)^2$$

Z_j standard coordinates of $T_{x_0}X \simeq \mathbb{R}^m$

$$\Rightarrow \quad F_u(\sqrt{u}D)(x_0, x_0) = F_u(\sqrt{u}\sqrt{L})(x_0, x_0)$$

<u>Prop</u>: $$e^{-uD^2}(x_0, x_0) = e^{-uL}(0,0) + O(e^{-c/u})$$

for $u \to 0$

<u>Def</u>: For $s \in C_0^\infty(X_0 = \mathbb{R}^m, E_0)$, $Z \in \mathbb{R}^m$, $u > 0$

$$(\delta_u s)(Z) := s(Z/\sqrt{u})$$

$$\begin{cases} \nabla_u^{E_0} := \delta_u^{-1}\sqrt{u}\,\nabla^{E_0}\delta_u & \text{connection} \\ L_u := \delta_u^{-1}\, u L\, \delta_u & \text{Laplacian} \end{cases}$$

Let $e^{-L_u}(Z, Z')$ be heat kernel of L_u w.r.t. $dv_{T_{x_0}X}(Z)$ (Euclidean volume form)

Then
$$(e^{-L_u} s)(Z) = \int_{\mathbb{R}^m} e^{-L_u}(Z, Z')\, s(Z')\, dv_{T_{x_0}X}(Z')$$
$$= (\delta_u^{-1} e^{-uL} \delta_u s)(Z)$$
$$= (e^{-uL} \delta_u s)(\sqrt{u} Z)$$
$$= \int_{X_0 \cong \mathbb{R}^m} e^{-uL}(\sqrt{u} Z, Z')\, \underbrace{(\delta_u s)(Z')}_{s(Z'/\sqrt{u})}\, \underbrace{dv_{X_0}(Z')}_{k(Z')\, dv_{T_{x_0}X}(Z')}$$

$k(Z') \in C^\infty(X_0, \mathbb{R}_{>0})$

$k(0) = 1$

$$= u^{m/2} \int_{\mathbb{R}^m} e^{-uL}(\sqrt{u} Z, \sqrt{u} Z')\, s(Z')\, k(\sqrt{u} Z')\, dv_{T_{x_0}X}(Z')$$

Prop:
$$e^{-L_u}(Z, Z') = u^{m/2} e^{-uL}(\sqrt{u} Z, \sqrt{u} Z')\, k(\sqrt{u} Z')$$
in particular, $e^{-L_u}(0, 0) = u^{m/2} e^{-uL}(0, 0)$

Step 3: Getzler rescaling on Clifford variables

$E_0 = S_0^{TX} \hat{\otimes} W_0 \qquad W_0 = W_{x_0} \quad S_0^{TX} = S^{T_{x_0}X}$

$$\mathrm{End}(E_0) = \underset{\substack{\shortparallel \\ C(T_{x_0}X)}}{\mathrm{End}(S^{T_{x_0}X})} \hat{\otimes} \mathrm{End}(W_0) \quad \text{over } \mathbb{C}$$

$$\overset{\text{symbol}}{=} \Lambda^{\bullet} T_{x_0}^* X \hat{\otimes} \mathrm{End}(W_0)$$

Def: For $u \in (0,1]$, $v \in TX_0$

$$c_u(v) := \frac{1}{\sqrt{u}} v^* \wedge - \sqrt{u}\, i_v \quad \circlearrowleft \Lambda^\bullet TX_0$$

Consider the change $c(v) \in \mathrm{End}(S_0^{TX})$

$$\longrightarrow \quad c_u(v) \in \mathrm{End}(\Lambda^\bullet T^* X_0)$$

Let $\tilde{L}_u$ be the operator obtained by replacing $c(v)$ by $c_u(v)$ in $L_u := \delta_u^{-1} u L \delta_u$

Recall: only the maximal degree contributes in $\mathrm{Tr}_s^{S^{TX}}[\cdot]$

Def: Note that $\mathrm{End}(\Lambda^\bullet T^* X_0) = \mathrm{Span}_{\mathbb{R}} \{ \tilde{e}^j \wedge, \ i_{\tilde{e}_j} \}_{j=1,\dots,m}$

$\{\tilde{e}_j\}$ ONB $\tilde{e}^j$ dual basis

For any $\alpha \in \mathrm{End}(\Lambda^\bullet T^* X_0)$

$$\alpha = \{\alpha\}^{top} \tilde{e}^1 \wedge \cdots \wedge \tilde{e}^m + \text{other terms}$$

Lemma:

$$\mathrm{Tr}_s^{E_0}[\underbrace{e^{-L_u}(0,0)}_{\in\, \mathrm{End}(S^{T_{x_0}X}) \hat{\otimes} \mathrm{End}(W_0) = \mathrm{End}(E_0)}] = (-2\sqrt{-1})^{\frac{m}{2}} u^{\frac{m}{2}} \mathrm{Tr}_s^{W_0}[\{\underbrace{e^{-\tilde{L}_u}(0,0)}_{\in\, \mathrm{End}(\Lambda^\bullet T^* X_0) \hat{\otimes} \mathrm{End}(W_0)}\}^{top}]$$

$$\mathrm{End}(\Lambda^\bullet T^* X_0) \hat{\otimes} \mathrm{End}(W_0) \xrightarrow{\{\cdot\}^{top}} \mathrm{End}(W_0)$$

Pf: $\alpha = a\, c(\tilde{e}_1) \cdots c(\tilde{e}_m) \in C(T_{x_0} X)$

$$c_u(\alpha) = a\, u^{-\frac{m}{2}} \tilde{e}^1 \wedge \cdots \wedge \tilde{e}^m + \text{other terms} \in \mathrm{End}(\Lambda^\bullet TX_0)$$

$$\{c_u(\alpha)\}^{top} = u^{-\frac{m}{2}} a$$

5

$$\mathrm{Tr}_s^{S^{TX}}[\alpha] = a\,\mathrm{Tr}_s^{S^{TX}}[c(\tilde{e}_1)\cdots c(\tilde{e}_m)] = (-2\sqrt{-1})^{\frac{m}{2}} a$$

#

Finally: $\tilde{L}_u \curvearrowright C^\infty(X_0, \Lambda^\bullet T^*X_0 \otimes W_0)$

and for $u \to 0$

$$\mathrm{Tr}_s[e^{-uD^2}(x_0,x_0)] \overset{\text{step 2}}{=} u^{-\frac{m}{2}} \mathrm{Tr}_s[e^{-uL}(0,0)] + O(e^{-c/u})$$

$$= u^{-\frac{m}{2}} \mathrm{Tr}_s[e^{-L_u}(0,0)] + O(e^{-c/u})$$

L_u conjugate to uL

$$= (-2\sqrt{-1})^{\frac{m}{2}} \mathrm{Tr}_s^{W_0}[\{e^{-\tilde{L}_u}(0,0)\}^{top}] + O(e^{-c/u})$$

Next goal: work out the limit $\tilde{L}_u$ as $u \to 0$

Thm:

$$\tilde{L}_u \to \tilde{L}_0 := -\sum_j \left[\frac{\partial}{\partial Z_j} + \frac{1}{4}\langle R^{TX}_{x_0} Z, \frac{\partial}{\partial Z_j}\rangle\right]^2 + R^W_{x_0}$$

local model Laplacian

$\tilde{L}_0 \curvearrowright C^\infty(\mathbb{R}^m, \Lambda^\bullet T^*\mathbb{R}^m \hat{\otimes} W_{x_0})$

pf: ①

$$\Gamma_Z^{S_0^{TX}} := \frac{1}{4}\sum_{jk} \langle \Gamma_Z^{TX_0} \tilde{e}_j, \tilde{e}_k\rangle_{g^{TX}} c(\tilde{e}_j) c(\tilde{e}_k)$$

$c(\tilde{e}_j) \in \mathrm{End}(S_0^{TX})$

along the path $\gamma: [0,1] \ni s \mapsto sZ \in X_0$, $\|Z\| < 2\varepsilon$

$$\nabla_{\dot\gamma}^{S_0^{TX}} c(\tilde{e}_j) = c(\nabla_{\dot\gamma}^{TX_0} \tilde{e}_j)$$

$$= c(\nabla_{\dot\gamma}^{TX} \tilde{e}_j) = 0$$

⑥

$$\Rightarrow \quad c(\tilde{e}_j) = c(e_j)_{x_0} \in \mathrm{End}(S_0^{TX})$$
on $B^{X_0}(0, 2\varepsilon)$

<u>Lemma</u>: $\Gamma^{TX}_Z = \frac{1}{2} R^{TX}_{x_0}(Z, \cdot) + O(\|Z\|^2)$
for $Z \in X_0$ near 0

Pf: $\nabla^{TX} = d + \Gamma^{TX}$

$$R^{TX} = d\Gamma^{TX} + \Gamma^{TX} \wedge \Gamma^{TX}$$

define the vector field $\mathcal{R} = \sum_j Z_j \tilde{e}_j = \sum_j Z_j \frac{\partial}{\partial Z_j}$

$\nabla^{TX} = d + \Gamma^{TX}$ via parallel transport e_j along path $s \mapsto sZ$ in $B^{X_0}(0, 2\varepsilon)$

$$\rightsquigarrow \quad \iota_{\mathcal{R}} \Gamma^{TX} \equiv 0 \quad \text{near } 0$$

$$\mathcal{L}_{\mathcal{R}} \Gamma^{TX} = [d, \iota_{\mathcal{R}}] \Gamma^{TX} = \iota_{\mathcal{R}} d\Gamma^{TX}$$
$$= \iota_{\mathcal{R}} (d\Gamma^{TX} + \Gamma^{TX} \wedge \Gamma^{TX})$$
$$= \iota_{\mathcal{R}} R^{TX}$$

$$\mathcal{L}_{\mathcal{R}} dZ^j = dZ^j$$

$\Gamma^{TX}_{Z=0}(e_j) = 0$

$$\mathcal{L}_{\mathcal{R}} \Gamma^{TX} = \mathcal{L}_{\mathcal{R}} \Big(\sum_j \Gamma^{TX}(e_j) dZ^j \Big)$$
$$= \sum_j \Big(\mathcal{L}_{\mathcal{R}} \Gamma^{TX}(e_j) + \Gamma^{TX}(e_j) \Big) dZ^j$$
$$= \sum_j R^{TX}(\mathcal{R}, e_j) dZ^j$$

$$\Rightarrow \Big(\sum_k Z_k \frac{\partial}{\partial Z_k} \Gamma^{TX}(e_j) \Big)_Z + \Gamma^{TX}_Z(e_j) = R^{TX}_Z(Z, e_j)$$

Taking Taylor expansion at $Z = 0$: $\quad 2 \frac{\partial}{\partial Z_k} \Gamma^{TX}_0(e_j) = R^{TX}_0(e_k, e_j)$

$$\Rightarrow \quad \Gamma^{TX}_{Z}(e_j) = \sum_j \frac{\partial}{\partial z_k}\Gamma^{TX}_{0}(e_j)\, z_k + O(\|z\|^2)$$

$$= \frac{1}{2}\sum_j \underbrace{z_k R^{TX}_0(e_k}_{R}, e_j) + O(\|z\|^2)$$

#

$$L_u = \delta_u^{-1} u L \delta_u$$

$$= \delta_u^{-1} u \Big(-\sum_{k,l} g^{kl}_Z \big(\nabla^{E_0}_{e_k}\nabla^{E_0}_{e_l} - \nabla^{E_0}_{\nabla^{TX}_{e_k} e_l}\big) + \frac{r^X_Z}{4} + \frac{1}{2}\sum_{k,l} R^{W_0}(\tilde{e}_k, \hat{e}_l)\, c(\tilde{e}_k)\, \hat{c}(e_l)\Big)\delta_u$$

$$= -\sum_{k,l} g^{kl}_{\sqrt{u}Z}\Big(\nabla^{E_0}_{u,e_k}\nabla^{E_0}_{u,e_l} - \sqrt{u}\,\nabla^{E_0}_{u,\nabla^{TX}_{e_k}e_l}\Big)$$

$$+ \frac{u\, r^X(\sqrt{u}Z)}{4} + \frac{1}{2} u R^{W_0}_{\sqrt{u}Z}(\tilde{e}_k, \tilde{e}_l)\, c(e_k)_{x_0}\, c(e_l)_{x_0}$$

Inside $\nabla^{E_0}_u = \nabla^{S^{TX}_0}_u + \nabla^{W_0}_u$

$\uparrow$ Clifford variables

$\tilde{\nabla}^{E_0}_u$ = replacing $c(v)$ in ∇^E_u by $c_u(v)$

connection on $C^\infty(X_0, \Lambda^\cdot T^*X_0 \hat{\otimes} W_0)$

<u>Lemma</u>: ① $\tilde{\nabla}^{E_0}_u = d + \frac{1}{4}\langle R^{TX}_{x_0} Z, \cdot\rangle_{g^{TX}_{x_0}} + O(\sqrt{u})$ as $u \to 0$.

② $\frac{1}{2}\sum_{k,l} u R^{W_0}_{\sqrt{u}Z}(\tilde{e}_k, \tilde{e}_l)\, c_u(e_k)\, c_u(e_l) = R^{W_0}_{x_0} + O(\sqrt{u})$

Pf: $\tilde{\nabla}^{E_0}_u = \delta_u^{-1}\sqrt{u}\,\tilde{\nabla}^{E_0}\delta_u$

$$= d + \delta_u^{-1}\sqrt{u}\,\tilde{\Gamma}^{S^{TX}_0}_Z\,\delta_u + \sqrt{u}\,\Gamma^{W_0}_{\sqrt{u}Z}$$

$$= d + \frac{\sqrt{u}}{4}\sum_{j,k}\langle \Gamma^{TX}_{\sqrt{u}Z}\tilde{e}_j, \tilde{e}_k\rangle_{\sqrt{u}Z}\, c_u(e_j)\, c_u(e_k)$$

$$= d + \frac{\sqrt{u}}{8} \sum_{j,k} \langle R^{TX}_{x_0}(\sqrt{u} Z, \cdot)\tilde{e}_j, \tilde{e}_k \rangle_{\sqrt{u} Z} \left(\frac{1}{u} e^j \wedge e^k + O\left(\frac{1}{\sqrt{u}}\right)\right) + O(\sqrt{u})$$

$$= d + \frac{1}{8} \sum_{j,k} \langle R^{TX}_{x_0}(Z, \cdot) e_j, e_k \rangle_{x_0} e^j \wedge e^k + O(\sqrt{u})$$

$$= d + \frac{1}{8} \sum_{j,k} \langle R^{TX}_{x_0}(e_j, e_k) Z, \cdot \rangle_{x_0} e^j \wedge e^k + O(\sqrt{u}).$$

$$\tilde{L}_u = \underbrace{-\sum_k \left(\frac{\partial}{\partial Z_k} + \frac{1}{4} \langle R^{TX}_{x_0} Z, e_k \rangle\right)^2 + R^{W_0}_{x_0}}_{\tilde{L}_0} + O(\sqrt{u})$$

<u>Thm</u>: $\exp(-\tilde{L}_u)(0,0) \xrightarrow[u \to 0]{} \exp(-\tilde{L}_0)(0,0)$

<u>Prop</u> (*): $\exp(-\tilde{L}_0)(0,0) = (4\pi)^{-m/2} \det{}^{1/2}\left(\frac{R^{TX}_{x_0}/2}{e^{R^{TX}_{x_0}/2} - e^{-R^{TX}_{x_0}/2}}\right) \exp(-R^{W}_{x_0})$

$$\in \Lambda^\bullet T^*_{x_0} X \otimes \mathrm{End}(W_0)$$

<u>End of the step 3</u>:

$$\mathrm{Tr}_s^{E_{x_0}}[e^{-uD^2}(x_0, x_0)] = (-2\sqrt{-1})^{m/2} \mathrm{Tr}_s^{W_0}\left[\{\exp(-\tilde{L}_u)(0,0)\}^{top}\right] + O(e^{-c/u})$$

$$\xrightarrow{u \to 0} (-2\sqrt{-1})^{m/2} \mathrm{Tr}_s^{W_0}\left[\{\exp(-\tilde{L}_0)(0,0)\}^{top}\right]$$

$$\mathrm{Tr}_s^{E_{x_0}}[e^{-uD^2}(x_0, x_0)]\, dv(x_0)$$

$$\xrightarrow[u \to 0]{} \Bigg(\underbrace{\left(\frac{-2\sqrt{-1}}{4\pi}\right)^{\frac{m}{2}} \mathrm{Tr}_s^{W_0}\left[\exp(-\tilde{L}_0)(0,0)\right]}_{\hat{A}(TX, \nabla^{TX})_{x_0} \underbrace{ch^{E/S}_{x_0}(E, \nabla^E)}_{\mathrm{Tr}_s^{W_0}\left[\exp\left(-R^W_{x_0}/2\pi\sqrt{-1}\right)\right]}} \Bigg)^{max}$$

#

An explanation to Prop $(*)$: Mehler's formula.

Let $\mathcal{A}$ be a commutative ring

$$R \in M_r(\mathcal{A})$$

$$R = (R_{ij})_{r\times r} \qquad R_{ij} \in \mathcal{A}$$

Suppose $R_{ij} = -R_{ji}$

Consider the harmonic oscillator on $\mathbb{R}^m \ni Z$

$$H = -\sum_j \left(\frac{\partial}{\partial Z_j} + \frac{1}{4} \sum_k R_{kj} Z_k \right)^2$$

Then $\left(\frac{\partial}{\partial t} + H \right) P_t(Z) = 0$ has a solution

$$P_t(Z) = \frac{1}{(4\pi t)^{m/2}} \det{}^{1/2}\left(\frac{tR/2}{e^{tR/2} - e^{-tR/2}} \right) \exp\left(-\frac{1}{4t} \left\langle Z \,\middle|\, \frac{tR}{2} \coth \frac{tR}{2} \,\middle|\, Z \right\rangle \right)$$

Pf : on $\mathbb{R}^2$ fix $\lambda \in \mathbb{R}$

$$H = -\left(\frac{\partial}{\partial Z_1} + \frac{\lambda}{4} Z_2 \right)^2 - \left(\frac{\partial}{\partial Z_2} - \frac{\lambda}{4} Z_1 \right)^2$$

Solution to $\left(\frac{\partial}{\partial t} + H_Z \right) P_t(Z_1, Z_2) = 0$

$$P_t(Z_1, Z_2) = \frac{1}{4\pi t} \frac{t\lambda/2}{\sinh(t\lambda/2)} \exp\left(-\frac{t\lambda}{2} \coth\left(\frac{t\lambda}{2}\right) \|Z\|^2/4t \right).$$

#